Coxeter groups and Kazhan-Lusztig theory in GAP

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- No support for multiparameter case

Coxeter groups

Definition

(W, S) is called a Coxeter system or Coxeter group iff

$$W = \langle S | \forall s, t \in S : (st)^{m_{st}} = 1 \rangle$$

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Examples include

- $Sym(n)$ with $S = \{ (12), (23), \ldots, (n-1, n) \}$
- dihedral groups
- Binary icosahedral group $2 \cdot Alt(5)$
- Weyl groups of algebraic groups, Lie groups, Kac-Moody groups, groups with BN-pair, ...

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  - Indecomposable: A_n, B_n $(= C_n)$, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(m)$

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- Left cells for finite groups

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- What else?